

LIMITED POWER BURSTS IN DISTRIBUTED MODELS OF NUCLEAR REACTORS

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Distributed kinetic models, especially the diffusion-approximation equations, are widely employed for analyzing reactor dynamics. Equations of this type are used for analyzing the stability of the stationary state, investigating the boundedness of the solutions, calculating the self-excited oscillations of the reactor power, etc. Among these problems the question of the boundedness of reactor power bursts is of practical importance.

It is shown in [1, 2] that when reactor control is lost at low power the change in the neutron flux can acquire the character of a burst, accompanied by a significant release of heat in the active zone. These processes were investigated in [1, 2] by the method of differential inequalities based on a lumped model of the kinetics. In [3] the method of separation of motions into fast and slow, followed by analysis of the geometry of the surface of the slow motions, was employed for these purposes.

We consider below the nonlinear equation of single-group diffusion approximation

$$\begin{aligned}
 l \frac{\partial \Phi}{\partial t} &= M^2 \nabla^2 \Phi + \Phi f(\sigma, \delta k_0) + \sum_{i=1}^6 \beta_i (\Phi_i - \Phi); \\
 \frac{\partial \Phi_i}{\partial T} &= \lambda_i (\Phi - \Phi_i), \quad i = 1, 2, \dots, 6; \\
 \frac{\partial x}{\partial t} &= Ax + \alpha \Phi, \quad \sigma = b^T x,
 \end{aligned} \tag{1}$$

with the boundary condition

$$\Phi + \alpha (\nabla \Phi, n) = 0 \tag{2}$$

on Γ . Here Γ is a closed convex surface, bounding the volume of the active zone Ω ; A is a matrix whose coefficients depend on r (the spectrum of A lies to the left of the half-plane); a is a vector whose components depend on r ; and, r is the radius vector. Physically, $\Phi = \Phi(r, t)$ is the neutron flux (in relative units); $f(\sigma, \delta k_0) = k_\infty - 1$; k_∞ is the multiplication factor; λ_i and β_i are the decay constant and fraction of radiators of the i -th group of delayed neutrons, respectively; l is the lifetime of prompt neutrons; M is the migration length; and, δk_0 is a parameter.

Let the transfer coefficient $\chi(p) = b^T (pI - A)^{-1} a > 0$ for $r \in \Omega$. We choose the function $f(\sigma, \delta k_0)$ in the form

$$f(\sigma, \delta k_0) = \delta k_0 + 4\delta k_0 \sigma (1 - \sigma) \tag{3}$$

and we show that bounded power bursts can occur in the system (1)-(3).

Bifurcation Diagram. The dependence of the solutions (in particular, the stationary solutions) on a parameter is called a bifurcation diagram. For the system (1)-(3) the main parameter determining the behavior of the system is δk_0 , and the bifurcation diagram of stationary solutions is determined by the dependence of σ or Φ on δk_0 (control parameter).

In [2] it was established that the system of equations of reactor kinetics can have a solution of the burst type, if its bifurcation diagram has the form shown qualitatively in Fig. 1. We shall investigate the type of bifurcation diagram of the stationary solutions for the system (1)-(3).

The stationary solutions of the system (1)-(3) are found from the nonlinear boundary-value problem

$$\begin{aligned}
 M^2 \nabla^2 \Phi + \Phi f(\chi(0)\Phi, \delta k_0) &= 0; \\
 \sigma &= \chi(0)\Phi, \quad \Phi = \Phi_r,
 \end{aligned} \tag{4}$$

where $\chi(0) = -b^T A^{-1} a > 0$, and the boundary conditions are determined by the expression (2).

In order to investigate the stability of the zeroth solution we associate to the boundary-value problem (4) the eigenvalue problem of the linearized operator

$$\begin{aligned} M^2 \nabla^2 \psi + (\delta k_0 - \beta) \psi + \sum \beta_i \psi_i &= p \psi; \\ \lambda_i (\psi - \psi_i) &= p \psi_i; \\ Ax + \alpha \psi &= p \psi \end{aligned} \quad (5)$$

with the boundary condition (2), if we set in it $\Phi \equiv \psi$. It is easy to show that the spectrum of the operator (5) for $\delta k_0 < \delta \bar{k}_0$ lies in the left-hand half-plane. At the point $\delta k_0 = \delta \bar{k}_0$ the eigenvalue $p_1 = 0$, and therefore the stationary solution $\Phi \equiv 0$ becomes unstable.

We shall seek a nontrivial solution of Eq. (4) in a neighborhood of the point $\delta k_0 = \delta \bar{k}_0$. Confining our attention in Eq. (4) to infinitesimals of second order, we have

$$M^2 \nabla^2 \Phi + \Phi (\delta k_0 + 4 \delta k_A \chi(0) \Phi) = 0$$

with the boundary condition (2). We set $\Phi = C \psi_1$, where ψ_1 is the eigenfunction of the operator (5) that corresponds to the eigenvalue p_1 , and from the condition that the residual be orthogonal to the function ψ_1 we find

$$C \left(lp_1 + \beta - \sum \frac{\lambda_i \beta_i}{\lambda_i + p_1} \right) \int_{\Omega} \psi_1^2 d\Omega + C^2 4 \delta k_A \chi(0) \int_{\Omega} \psi_1^3 d\Omega = 0.$$

Besides the zero solution this equation has the solution

$$C = - \frac{\left(lp_1 + \beta - \sum \frac{\lambda_i \beta_i}{\lambda_i + p_1} \right) \int_{\Omega} \psi_1^2 d\Omega}{4 \delta k_A \chi(0) \int_{\Omega} \psi_1^3 d\Omega}.$$

Since $\Phi > 0$, a nontrivial solution exists for $\delta k_0 < \delta \bar{k}_0$.

We now show that for δk_0 less than δk_0^* the system (1)-(3) does not have nontrivial stationary solutions. We employ the boundedness of the function $\varphi(\sigma) = 4 \delta k_A \sigma (1 - \sigma) \leq \delta k_A$, and we consider the linear system

$$\begin{aligned} l \frac{\partial \Phi}{\partial t} &= M^2 \nabla^2 \Phi + \Phi (\delta k_0 + \delta k_A) + \sum_{i=1}^6 \beta_i (\Phi_i - \Phi); \\ \frac{\partial \Phi_i}{\partial t} &= \lambda_i (\Phi - \Phi_i), \quad i = 1, 2, \dots, 6; \\ \frac{\partial x}{\partial t} &= Ax + \alpha \Phi, \quad \sigma = b^T x. \end{aligned} \quad (6)$$

For $\delta k_0 < \delta k_0^* = \delta \bar{k} - \delta k_A$ the spectrum of the operator (6) lies in the left-hand half-plane. Since the operator of displacement along trajectories for this system is monotonic [4], all solutions of the system (1) are bounded from above by the solutions of the system (6) and therefore as $t \rightarrow \infty$ they approach $\Phi \equiv 0$.

Finally, we prove that all solutions of Eqs. (1)-(3) for physically realizable initial conditions and finite values of δk_0 are globally bounded. It is easy to show that feedback in the system (1)-(3) is sublinear [5], since there exist positive constants α and γ for which

$$f(\sigma, \delta k_0) \leq \alpha - \gamma \sigma \quad (7)$$

and, in addition, the transfer factor $\chi(p)$ for all $\omega \in [0, \infty]$ satisfies Welton's condition

$$-\gamma \operatorname{Re} \chi(i\omega) < 0. \quad (8)$$

When the inequalities (7) and (8) are satisfied, the solutions of the system (1)-(3) are bounded [5].

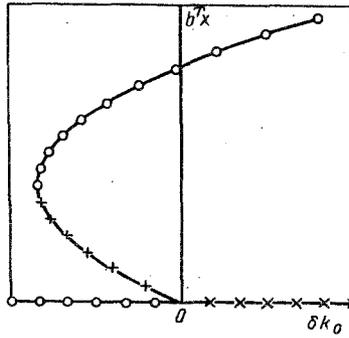


Fig. 1. Qualitative form of the bifurcation diagram.

Combining the results of our analysis of the system (1)-(3), it can be concluded that the bifurcation diagram of the stationary solutions does indeed have the form displayed qualitatively in Fig. 1, and therefore the system (1)-(3) has a solution of the burst type.

Estimate of the Solutions from Below. When control is lost at low power, the reactor cannot remain long near an unstable state of equilibrium, and the image point on the bifurcation diagram will move along the vertical straight line $\delta k_0 = \text{const}$ (see Fig. 1), passing through the corresponding states of equilibrium, upwards or downwards depending on the initial perturbation. The horizontal distance from a point on the straight line $\delta k_0 = \text{const}$ up to the bifurcation diagram will equal the reactivity of the reactor for fixed σ . If, in addition, $\delta k_A > \beta$ and $\delta k_0 \approx 0$, then the transient process has a slow initial stage, which transforms into a fast stage on instantaneous kinetics.

We shall study fast processes, in whose analysis the nonlinear feedback (3) can be replaced by linear feedback $f(\sigma, \delta k_0) = \delta k_0 - \gamma\sigma$, bounding it from above. In order to substantiate this substitution, we point out that the nonlinear section of the bifurcation diagram corresponding to small values of σ (or Φ) plays a significant role at the slow initial stage of the process and is insignificant for describing processes on instantaneous kinetics. We shall now find the conditions under which on the solution starting in an arbitrarily small neighborhood of the state of equilibrium, $\Phi(r, t)$ reaches a value not less than Φ_m , at some time $t = T$.

We now consider the Lyapunov functional [1, 5]

$$V = \int_{\Omega} \psi \left(l\Phi + \sum \beta_i \frac{\Phi_i}{p + \lambda_i} - x^T Hx \right) d\Omega,$$

where H is a symmetric matrix; p is a positive number which we shall define below; $\psi \in C^1$ on $\bar{\Omega}$, $\psi \neq 0$ in Ω and satisfies the boundary condition $\psi + \alpha(\nabla\psi, n) = 0$ on the boundary Γ of the region Ω .

In accordance with the system (1) the time derivative has the form $dV/dt = pV + W$, where

$$W = \int_{\Omega} \psi \{ M^2 \nabla^2 \Phi - \gamma\sigma\Phi + [\delta k_0 - G(p)] - [x^T (A^T H + HA - pH) x + 2a^T Hx\Phi] \} d\Omega; \tag{9}$$

$$G(p) = lp + \sum \frac{p\beta_i}{p + \lambda_i}.$$

We now consider the auxiliary boundary-value problem for the function ψ

$$\nabla^2 \psi + \lambda\psi = 0 \tag{10}$$

with the boundary condition $\psi + \alpha(\nabla\psi, n) = 0$ for $r \in \Gamma$. Multiplying Eq. (10) by ΦM^2 and combining it with the relation (9), we obtain

$$W = \int_{\Omega} \psi \{ -\gamma\sigma\Phi + [\delta k_0 - \lambda M^2 - G(p)] \Phi - [x^T (A^T H + HA - pH) x + 2a^T Hx\Phi] \} d\Omega.$$

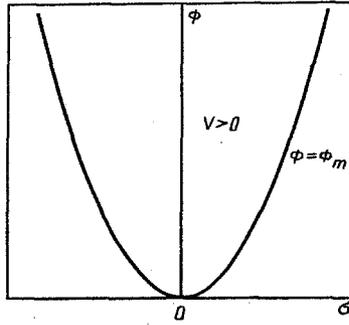


Fig. 2. Qualitative form of the set $V > 0$.

Here we employed the relation

$$\int_{\Omega} (\psi \nabla^2 \Phi - \Phi \nabla^2 \psi) d\Omega = \int_{\Gamma} [(\nabla \Phi, n) \psi - (\nabla \psi, n) \Phi] dS = 0.$$

Using the inequality $\Phi \leq \Phi_m$, we obtain

$$W \geq \int_{\Omega} \psi \{-\gamma \sigma \Phi + \rho \Phi^2 - [x^T(A^T H + HA - \rho H)x + 2a^T H x \Phi]\} d\Omega,$$

where $\rho > 0$, $\Phi \rho = \delta k_0 - \lambda M^2 - G(p)$.

Next, using Yakubovich's matrix inequalities [6], we arrive at the particular condition for determining ρ and the matrix equations for determining H :

$$\begin{aligned} \rho &\geq \gamma \operatorname{Re} \chi; \rho \geq 0; \chi = b^T \left(\frac{\rho}{2} I - A + i\omega I \right)^{-1} a; \\ -\rho H + HA + A^T H &= -\frac{1}{\rho} g g^T; \\ Ha + \gamma \frac{b}{2} &= -g, \end{aligned}$$

where I is a unit matrix.

Thus Φ_m can be found from the inequalities

$$\begin{aligned} \Phi_m \rho &= \delta k_0 - \lambda M^2 - G(p); \\ \rho &\geq \gamma \operatorname{Re} \chi; \omega \in [0, \infty). \end{aligned} \quad (11)$$

We arrive finally at the differential inequality $dV/dt \geq pV$, whence it follows that $V(t) \geq V(0) \exp(pt)$. The condition $V(0) > 0$ singles out of the space of states the set of initial conditions for which the solution grows without limit, while $\Phi \leq \Phi_m$. This means that there exists a time $t = T$ at which $\Phi \geq \Phi_m$.

The possible values of p are found from the inequalities $0 \leq G(p) \leq \delta k_0 - \lambda M^2$. Since $|\chi(p/2 + i\omega)| \rightarrow 0$, as $p \rightarrow \infty$, for large values of δk_0 (when the range of possible values of p is large) Φ_m is large. The region $V > 0$ is displayed qualitatively in Fig. 2.

We now consider the simplest example of feedback in order to illustrate the calculation of Φ_m . Using the expression (11) we have

$$\rho \geq \bar{\rho} \left(\frac{\rho}{2} \right) = \sup_{\omega} \left[0, \gamma \operatorname{Re} \chi \left(\frac{\rho}{2} + i\omega \right) \right];$$

$$\Phi_m = \frac{\delta k - \lambda M^2 - G(p)}{\bar{\rho}(p/2)}.$$

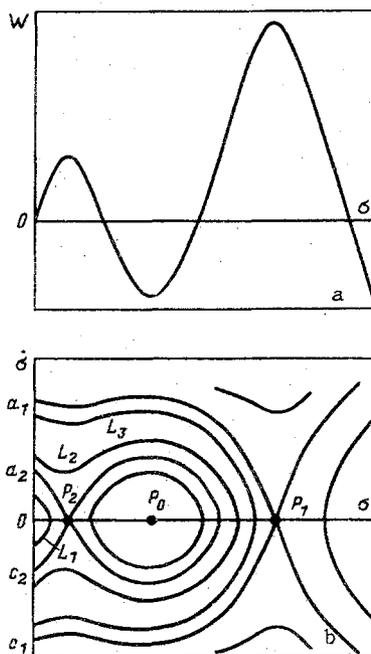


Fig. 3. $W(\sigma)$ for $\bar{Y} < Y_*$ (a) and the corresponding phase portrait of the system (13) b).

To each value of p satisfying the condition $0 \leq G(p) \leq \delta k_0 - \lambda M^2$, there is associated a number Φ_m and a region in phase space. Among these values of p there is a value of p that gives to Φ_m a maximum value, which we designate by $\bar{\Phi}_m$.

For definiteness, let $\chi(p) = 1/(1 + Tp)$, $\delta k_0 = 5\beta$, $\gamma = 2\beta$, $\lambda M^2 = 1.5\beta$, and $T = 10$ sec. Evidently, $G(p) \leq \beta + lp$, $\rho = \gamma/(1 + Tp/2)$ and therefore

$$\Phi_m \geq \left(1 + \frac{Tp}{2}\right) \frac{(\delta k_0 - \lambda M^2 - \beta - lp)}{\gamma}.$$

The maximum value of the right-hand side is reached for $lp = (\delta k_0 - \lambda M^2 - \beta - 2l/T)/2$. For $l/\beta = 10^{-2}$ sec and the other parameters having the values indicated above, $\bar{\Phi}_m > 10^3/3$. If T is set equal to 1 sec, then $\bar{\Phi}_m$ will be smaller, but still large: $\bar{\Phi}_m > 10^2/3$.

Thus we have verified for the simplest example that in phase space there exists a region $V > 0$, adjoining the zero state of equilibrium, such that when the image point enters this region the neutron flux increases (with exponent p) up to values many times greater than the stationary (nominal) value $\Phi = 1$.

Surface of Slow Motions. In [3] a method for separating motions into fast and slow was employed in order to investigate discontinuous solutions in the lumped model of reactor kinetics. We shall show that this approach can be extended to the case of distributed models.

In order to simplify the calculations we specify feedback by an equation of the form $d\sigma/dt = \lambda(\Phi - \sigma)$, where $1/\lambda$ is the heating time of the fuel tablets, and we describe the delayed neutrons by a single group with a time constant equal to the time constant of heating of the fuel tablets ($1/\lambda_i = 1/\lambda \cong 10$ sec for oxide fuel). Then, eliminating Φ and Φ_i from the first equation of the system (1), we arrive at the following system:

$$\begin{aligned} \frac{l}{\lambda} \frac{\partial \sigma}{\partial t} + l\sigma + \frac{\beta}{\lambda} \sigma - \frac{M^2}{\lambda} \nabla^2 \sigma - \frac{1}{\lambda} \int_0^\sigma f(\xi, \delta k_0) d\xi &= Y; \\ \frac{\partial Y}{\partial t} &= \sigma f(\sigma, \delta k_0) + M^2 \nabla^2 \sigma, \end{aligned} \quad (12)$$

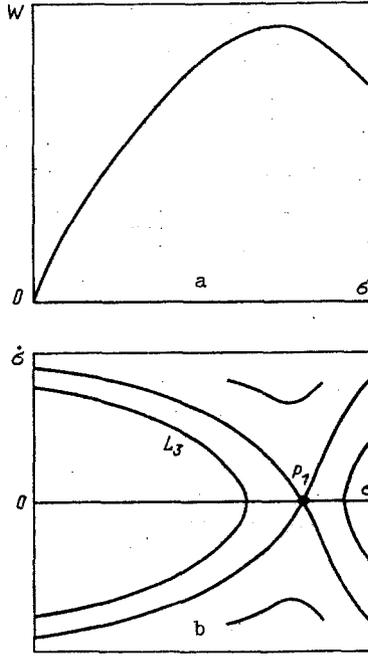


Fig. 4. Same as Fig. 3, but for $\bar{Y} > \bar{Y}_*$.

where $\beta = \sum_{i=1}^6 \beta_i$, and the boundary conditions are determined by the expression (2), in which we must set $\Phi \equiv \sigma$. In the system (12) the parameter l/λ in front of the derivative is small, so that all motions in phase space can be divided into fast motions, occurring along the straight lines $Y = \text{const}$, and slow motions, whose trajectories lie on the surface $Y = F(\sigma)$. Setting $l/\lambda = 0$, we find the surface of slow motions

$$Y = \frac{\beta}{\lambda} \sigma - \frac{M^2}{\lambda} \nabla^2 \sigma - \frac{1}{\lambda} \int_0^\sigma f(\xi, \delta k_0) d\xi.$$

We now consider the one-dimensional problem ($\nabla^2 \rightarrow \partial^2/\partial x^2$). Using the expression (3) for $f(\sigma, \delta k_0)$, we obtain

$$\frac{d^2 \sigma}{dx_n^2} - s\sigma + \frac{\sigma^2}{2} - \frac{\sigma^3}{3} + \bar{Y} = 0, \quad (13)$$

where $x_n = x\sqrt{\delta k_A/M^2}$; $s = (\beta - \delta k_0)/4\delta k_A > 0$; $\bar{Y} = Y\lambda/4\delta k_A$. Integrating the condition (13), we arrive at the following equation: $(1/2)(d\sigma/dx_n)^2 + W(\sigma) = 0$, where $W = -s\sigma^2/2 + \sigma^3/6 - \sigma^4/12 + \bar{Y}\sigma$.

Figure 3 displays the qualitative form of the function $W(\sigma)$ for quite small values of \bar{Y} and the corresponding phase portrait of the system (13). As the parameter \bar{Y} increases the states of equilibrium P_0 and P_2 approach one another and for some $\bar{Y} = \bar{Y}_*$ they merge. The phase portrait of the system (13) for $\bar{Y} > \bar{Y}_*$ is displayed in Fig. 4. Only trajectories in the phase plane which satisfy the boundary conditions are physically meaningful. It is obvious that the separatrices of the saddles $a_1 P_1 c_1$ and $a_2 P_2 c_2$ correspond to stationary soliton-like solutions of the initial system, determined on an infinite straight line. For a system given in a bounded region the trajectories L_1, L_2, L_3 (see Fig. 3) correspond to soliton-type solutions. In addition, the larger the region, the closer the trajectories, satisfying the boundary conditions, are to the separatrices.

Thus for $\bar{Y} < \bar{Y}_*$ a system given in a finite region has three stationary solutions (L_1, L_2, L_3) the amplitudes of these solutions being significantly different. For $\bar{Y} > \bar{Y}_*$ only the solution with the greatest amplitude remains. A similar scenario occurs as \bar{Y} decreases (for $\bar{Y} = \bar{Y}_{**} < \bar{Y}_*$ the states of equilibrium P_0 and P_1 merge). On the basis of the theory of bifurcations (see, for example, [7]) and what we have said above it can be concluded that two stationary solutions (L_1 and L_3) should be stable and the third solution (L_2) should be unstable.

Thus a qualitative analysis shows that the surface of slow motions of the system (12) has points of a fold-type catastrophe. As the system moves along the slow-motion surface (this corresponds to a change in \bar{Y}) it can reach one such point, after which it hops abruptly out of the region where solutions corresponding to small heat release exist into the region where such solutions do not exist. This transition will correspond to rapid growth of heat release in the active zone, i.e., a burst.

REFERENCES

1. E. F. Sabaev, "Explosive, unbounded, and globally bounded solutions of the equations of reactor kinetics with feedbacks on reactivity," *Voprosy Atomnoi Nauki i Tekhniki Ser. Fizika i Tekhnika Yadernykh Reaktorov*, No. 2(39), 10-19 (1984).
2. E. F. Sabaev, "Transient processes in boiling-water channel reactors with loss of control at low power levels," *Voprosy Atomnoi Nauki i Tekhniki Ser. Fizika i Tekhnika Yadernykh Reaktorov*, No. 1, 3-9 (1988).
3. M. V. Bazhenov, "Existence of discontinuous solutions of the equations of kinetics of a nuclear reactor with feedback," in: *Applied Problems in the Theory of Oscillations* [in Russian], Nizhegorod Univ. (1991), pp. 95-103.
4. E. F. Sabaev, *Comparison Systems for Nonlinear Differential Equations and Their Applications in Reactor Dynamics* [in Russian], Atomizdat, Moscow (1980).
5. Ya. V. Shevelev (ed.), *Dynamics of Nuclear Reactors* [in Russian], Énergoatomizdat, Moscow (1990).
6. F. R. Gantmakher and V. A. Yakubovich, "Absolute stability of nonlinear regulatable systems," in: *Proceedings of the Second All-Union Conference on Theoretical and Applied Mechanics* [in Russian], Nauka, Moscow (1965), No. 1, pp. 30-60.
7. V. S. Afraimovich, N. K. Gavrilov, V. I. Luk'yanov, and L. P. Shil'nikov, *Basic Bifurcations of Dynamical Systems* [in Russian], Gor'kii Univ., Gor'kii (1985).