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The "amplitude"-"phase" turbulence transition in a Ginzburg-Landau model as a critical phenomenon

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Spatial disorder in the large box, one-dimensional, complex Ginzburg-Landau problem is investigated quantitatively. The transition from phase to amplitude turbulence is studied in detail. This transition is described by the dimension of the space series, d_s, that estimates the number of normal (independent) medium perturbations forming the chaotic space series. It is found that at a critical point, d_s undergoes a jump whose value is universal, i.e. does not depend on the dimension of the system. Thus the number of perturbations in the medium grows anomalously near the transition point. This behavior is typical for critical phenomena.

1. The generalized (or complex) Ginzburg-Landau equation (CGLE)

$$\partial\tilde{a}/\partial t=\gamma\tilde{a}+(\mathrm{i}\delta-\rho)\tilde{a}|\tilde{a}|^2+(\mathrm{i}\kappa+\nu)\Delta\tilde{a} \qquad (1)$$

describes the wave dynamics of many different physical situations (e.g., shear hydrodynamic flows [1], chemical reactions in media with diffusion [2], water [3] and plasma [4] waves, etc.). Its universality has a very simple explanation. This equation can be obtained, independent of the physical origin of the medium or field, as a result of the transition to slow coordinates and time in all cases of a complex amplitude of quasi-monochromatic wave perturbations proportional to $\exp(-\mathrm{i}\omega t+\mathrm{i}kx)$. It provides the correct asymptotic behavior when the instability threshold is slightly exceeded in a narrow spectral interval.

The CGLE model is a convenient tool for the investigation of basic phenomena in the nonlinear dynamics of nonequilibrium media. The birth and subsequent evolution of spatio-temporal chaos (turbulence), both in the evolution of a regular initial field distribution and as a result of parameter variation, is one of the principal problems in this field [5].

Numerous computer and analytical investigations have shown that (depending on the parameter ratio) both weak turbulence, which is characterized only by irregular phase pulsations, and a regime of strong turbulence may be established in a two-dimensional CGLE system. Localized long-lived structures (spiral vortices) are typical for the regime of strong turbulence. The motion and interaction of these structures determine the properties of the two-dimensional turbulence of the wave field envelope [5-11]. Two-dimensional effects in a CGLE are extremely diverse and attractive. However, even in a one-dimensional model the spatio-temporal behavior of the complex amplitude has a great variety of nontrivial effects. Within a one-dimensional model there also exist well-pronounced localized structures whose interaction may lead to different dynamical and turbulent regimes. The investigation of one-dimensional systems is undoubtedly much simpler than that of two-dimensional systems. But even in one dimension much is yet to be understood. In particular, there are no answers to questions such as: What happens in the phase space of the system when the regime of phase turbulence is replaced by the regime of strong turbulence? What are the governing parameters of this transition and how is it to be described? Our paper is concerned with these problems. Apparently, the information on the dynamics of one-dimensional structures may be useful for the analysis of the properties of stochastic self-modulation in two- and three-dimensional systems.

The architecture of the remainder of this paper is as follows. Section 2 presents an analysis of the peculiarities of localized CGLE solutions in the form of kinks or "holes". These structures are useful in the construction of models for strong turbulence. In section 3 we propose to use the fractal dimension of the space series as a description of the transformation from various spatio-temporal regimes within CGLE. We also use it to investigate the properties of the "phase turbulence – strong turbulence" transition. Finally, the relation between the fractal dimension of the time series on the one hand, and the space series on the other is considered in section 4.

2. The one-dimensional system

$$\partial a/\partial t = Ra - a|a|^2 + \partial^2 a/\partial x^2 + \mathrm{i}\alpha\, \partial^2 a/\partial x^2 + \mathrm{i}\beta a|a|^2 \tag{2}$$

with the periodic boundary conditions

$$a(x+2\pi, t) = a(x, t) \tag{3}$$

is obtained from eq. (1), for solutions with a spatial period $L/2\pi$, by substituting

$$R = \gamma L^2/\nu, \quad \beta = \delta/\rho, \quad \alpha = \kappa/\nu, \quad a = \tilde{a}L(\rho/\nu)^{1/2} \tag{4}$$

and using new dimensionless time and space coordinates: $t = t_{\rm old}\nu/L^2$ and $x = x_{\rm old}/L$.

Note that the growth rate, R, also determines the space scale of modulation structures, l. Linear stability is compensated by nonlinear dissipation and diffusion, i.e.,

$$R \sim |a|^2 \sim l^{-2} \equiv |a^{-1}\,\partial^2 a/\partial x^2| \, . \tag{5}$$

With increasing growth rate, the characteristic amplitude of the appearing structures grows and their characteristic scale, l, diminishes.

Detailed analytical analysis of the behavior of the field $a(x, t)$ is possible only in limiting cases: in a conservative medium without diffusion ($\alpha, \beta \to \infty$) [#1] or in dissipative media without dispersion ($\alpha, \beta \to 0$).

[#1] In this case from (2) we have a nonlinear Schrödinger equation (NSE) where the spatial structures are envelope solitons whose amplitude A and velocity V may slowly vary under the action of weak amplification or dispersion [4–12].

When $\alpha = \beta = 0$, system (2) transforms into a nonlinear diffusion equation that can be written in a gradient form:

$$\frac{\partial a}{\partial t} = -\frac{\delta F}{\delta a^*}, \tag{6}$$

where

$$F = -\int [R|a|^2 - \tfrac{1}{2}|a|^4 - (\partial a/\partial x)^2]\, \mathrm{d}x$$

is a Lyapunov functional. Since the functional, $F(a)$, may only decrease along the trajectory ($\mathrm{d}F/\mathrm{d}t = -\int |\partial a/\partial t|^2\, \mathrm{d}x \leqslant 0$), the phase space of the dynamical system (2), (3) may contain only static attractors, i.e. equilibrium states. Consequently, the spatial distribution of the fields corresponding to these attractors may be only regular and is described by the following equation for a nonlinear complex oscillator: $\mathrm{d}^2a/\mathrm{d}x^2 = a(|a|^2 - R)$.

In particular a stationary kink,

$$a = R\, \mathrm{th}(x/\sqrt{2}) \, , \tag{7}$$

that corresponds to the boundaries between domains (i.e. the regions with a homogeneous complex amplitude whose phases differ by π) are of special interest. The amplitude profile in this solution contains a gap – a "hole" [13]. Such "hole" solutions persist for complex coefficients in the Ginzburg–Landau equation [14]. It is significant that one-dimensional structures ("holes") behave like the cores of two-dimensional structures (spirals).

The phenomenology of spatio-temporal chaos within a CGLE has been studied in ample detail. It has been found that if the parameter $p = \sqrt{\alpha\beta} > 1$ is not too high, then the spatially homogeneous distribution of the amplitude becomes unstable, and a regime that is referred to as "phase turbulence" is established [1,2]. Slow random walks of the phase, φ, of the wave field, $a = A\exp(\mathrm{i}\varphi)$, are typical for this regime, as long as the fluctuations of the amplitude, A, are small and "follow" the phase fluctuations. As the parameter p increases, the behavior of turbulence changes. Regions of abrupt phase variations appear and, respectively, there occur "holes" in the amplitude profile. It was shown in ref. [14] that as the parameter p increases, the "holes" grow in number, move (see ref. [15]), oscillate and interact with one another. The resulting developed "amplitude" turbulence makes a quite complicated picture.

It is important to understand the processes that occur in the phase space of the CGLE and to describe them in terms of nonlinear dynamics. It is also essential to find whether the characteristics of the "phase–amplitude" turbulence transition are universal. For the solution of this problem we employ a new approach to the analysis of the phase-to-amplitude turbulent transition. This approach is based on the calculation of the dimension of spatial chaos and on the analysis of the parameter dependence of the variation of this quantity.

3. The correlation dimension of the space series, d_s, was first introduced in ref. [16]. When calculating d_s we assume that the space series at a certain moment of time, t, may be considered to be generated by a finite-dimensional translational dynamical system, G_x.

In a one-dimensional medium, d_s is determined similarly to the correlation dimension of the time series, d_t, [17] employing the correlation integral. Assume that the dynamical system, G_x, in an m-dimensional phase space describes the trajectory $y(x)=\{a(x),\ a(x+\xi),\ a(x+2\xi),\ ...,\ a[x+(m-1)\xi]\}$. We will calculate the dimension using the correlation integral in a standard form:

$$R^m(\epsilon)=(M-m)^{-2}\sum_i\sum_j H(\|y_i-y_j\|-\epsilon)$$

$$=\langle M^i_\epsilon\rangle/M\,, \tag{8}$$

where $H(\epsilon)$ is the Heaviside function, y_i is a point in the m-dimensional phase space, M denotes the total number of points in the series, and M^i_ϵ, the number of points in the ϵ-neighborhood of the ith point. Because $R^m(\epsilon)\sim\epsilon^{d_s}$, the correlation dimension is determined as the ratio $\log[R^m(\epsilon)]/\log(\epsilon)$ for small enough ϵ. Note that correct calculation of the dimension, d_s, needs sufficiently long space series and a proper choice of ξ (see appendix). Definition (8) is analogous to the definition of the temporal correlation function as a time-averaged number of "fragments" of the series, with the distance between the "fragments" being smaller than ϵ. In a similar fashion and in analogy with the local dimension that is determined by averaging over time, the value of the correlation dimension of the space series, d_s, in m-dimensional enclosure space, is defined as the ratio $\log[R^m(\epsilon)]/\log(\epsilon)$ for sufficiently small ϵ.

The length of the space series is determined by the actual number of spatial structures. In the case of the boundary value problem (2), (3), the value $R\gg 1$ corresponds to a long space series. The dimensional characteristics of the turbulent regime that has been established as $t\to\infty$, in this case, depend neither on the boundary conditions nor on the length of the system.

4. We shall now investigate the transformations of the attractor that corresponds to the regime of established spatio-temporal chaos in a long system. To this end we will measure the dimensions d_s (for the

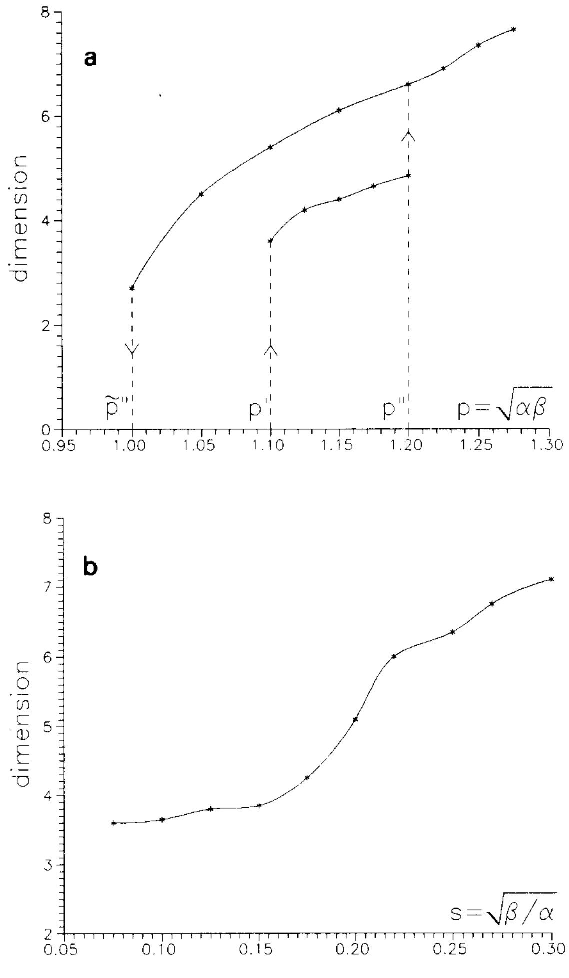

Fig. 1. Dimension of the space series for $R=10^4$; (a) against p for $s=1.15$; (b) against s for $p=3.0$.

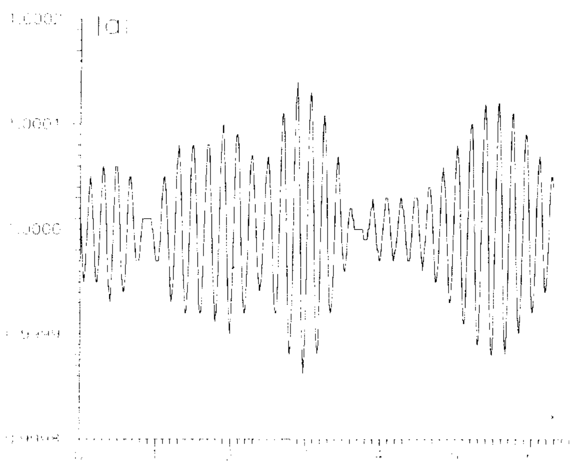

Fig. 2. Space amplitude variations for $R=10^4$, $\alpha=1.05$, $\beta=1.3$.

space series) and d_t (for the time series). The dependencies of fractal dimensions on the product $p=\sqrt{\alpha\beta}$ are given in fig. 1a for fixed $s=\sqrt{\beta/\alpha}$ [#2]. Of principal importance here are two phenomena: the jump in dimension when $p=p''$ and the hysteresis in the parameter region $p\in(\tilde{p}'', p'')$. To understand their origin we shall consider the variation of d_s and d_t with increasing p.

For small p the dimension is equal to zero, which corresponds to the regime of spatially homogeneous $e^{i\beta t}$-oscillations. When $p>p_1$, the dimension grows up to unity, which corresponds to the onset of a spatially periodic regime, while for $p=p_2$ a double frequency (2-D) regime is established. Further evolution of the system is related to the appearance of beats against the background of quasi-periodic oscillations (see fig. 2) and to the transition, when $p>p'$, to a chaotic regime. Note that for large values of the parameter R, the region on the plane of the parameters (α, β) that corresponds to periodic and quasi-periodic oscillations is very narrow and is not depicted in fig. 1 [#3].

A developed turbulent regime that sets in when

[#2] The combinations of the parameters α and β are chosen such as to move along the normal to the boundary of the regions of phase and amplitude turbulence (see below).

[#3] The large-scale structures whose size is comparable with a resonator length (a short system, small R) was studied earlier [18,19].

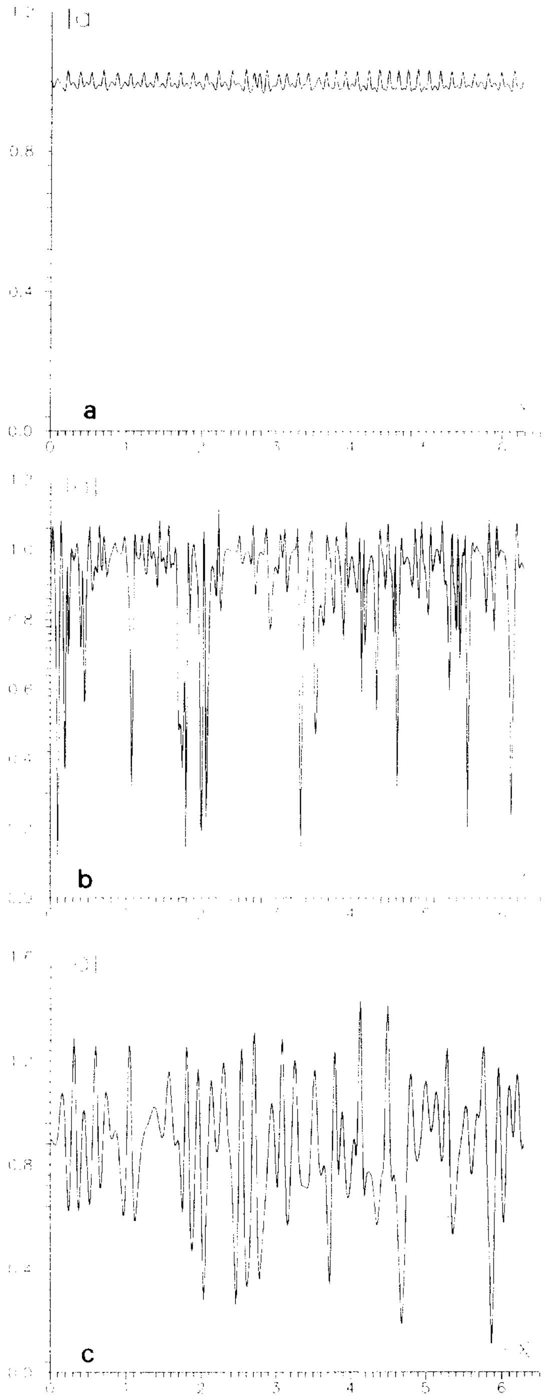

Fig. 3. Typical "snapshots" in turbulent regimes: (a) phase turbulence for $R=10^4$, $\alpha=1.0$, $\beta=1.3$; (b) amplitude turbulence for $R=10^4$, $\alpha=1.1$, $\beta=1.44$; (c) amplitude turbulence for $R=10^4$, $\alpha=12.0$, $\beta=0.75$.

$p \geqslant p'$ is phase turbulence: the amplitude has a value close to $|\alpha| = \sqrt{R}$ (fig. 3a) and its fluctuations adiabatically "follow" relatively smooth random walks of the phase. While moving over the parameter plane deep into the region of phase turbulence (with p increasing for $s=\text{const}$) up to $p=p''$, the stochastic set corresponding to phase turbulence gets more and more complicated; the dimension of the space series grows monotonically within the interval $3 < d_s \leqslant 5$.

Past the critical value of the parameter $p=p''$, we observe a qualitatively new behavior: the dimension of the space series increases abruptly up to $d_s \geqslant 6$. This is accompanied with strong oscillations in the spatial amplitude distribution – amplitude turbulence is established.

The spatial amplitude distribution in this regime may have different characteristic forms. When $\beta \gg 1$, the turbulence typically has narrow impulses ("holes") on the background of weak oscillations near the level $|a| = \sqrt{R}$ (fig. 3b). With increasing p, such holes grow in number past the critical point $p=p'$, which, evidently, leads to a further growth in the dimension of the space series. When $\beta \leqslant 1$, the number of "holes" is sufficiently large immediately past the threshold where amplitude modulation is established, consequently, the spatial picture is visualized as strong oscillations in the interval from zero to a certain maximum (fig. 3c).

In a reverse motion over the parameter plane hysteresis is observed: the amplitude turbulence is retained and does not change to the phase turbulence even when $p < p'$. Stepwise decrease of dimension occurs when $p = \tilde{p}''$. Thus, the initial conditions determine the regime – developed amplitude turbulence or phase turbulence – to be established in the region $(\tilde{p}'', p'')$.

A possible mechanism responsible for the phenomenon described above is as follows. The transition of the motion of the translational dynamical system, G_x, from one stochastic set in the phase space to another corresponds to a sharp change in the value of the space series dimension. The established turbulent regime corresponds to a strange attractor in the phase space. The phase transition occurs at the instant the separatrix manifold of the saddle periodic motion bounding the region of attraction of a low-dimensional ("phase") attractor merges with the latter which, as a result, ceases to be an attractor: now

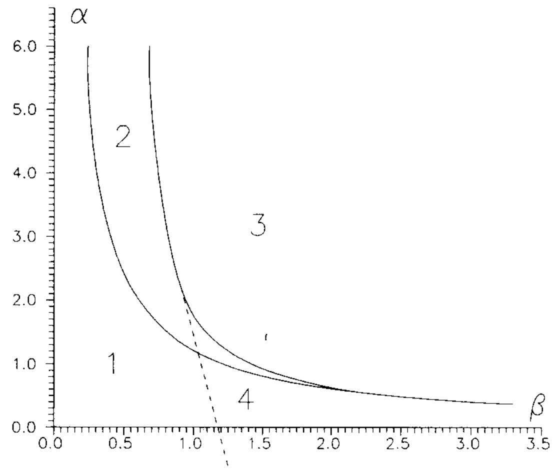

Fig. 4. (α, β)-plane for $R = 10^4$ with regions of different dynamics: (1) stable monochromatic homogeneous oscillations with amplitude $|a| = \sqrt{R}$; (2) phase turbulence; (3) amplitude turbulence; (4) region of hysteresis.

nearly all the trajectories tend to each other, "laminar" attractor. Both strange attractors – the phase and the amplitude ones – coexist in the hysteresis region in the phase space.

Partitioning of the parameter space (α, β) into regions of existence and stability of different spatially disordered regimes is presented in fig. 4 [#4]. In region 1 the homogeneous amplitude field is stable. Phase turbulence appears from small initial disturbances in region 2, and amplitude turbulence exists in region 3. The broken curve bounds the region of hysteresis (region 4 in fig. 4), where the solution depends on initial conditions. Note that this region stretches to the negative halfplane of the parameter $\alpha < 0$, where the space series (static lattices of randomly located holes) are established. There is no hysteresis in region 2 (outside region 4). The transition from phase to amplitude turbulence occurs here not by a jump but gradually in a finite interval: the intensity of phase fluctuations increases near the boundary of region 2. The rapid growth of the dimension is observed in this interval (fig. 1b), in connection with the appearance of holes.

[#4] A similar partitioning of parameter space into regions of different behavior of the correlation function (which is determined by the presence or absence of "holes") was performed in ref. [20].

5. Of principal importance is the problem of the relationship between different quantities describing turbulence: the number of structures, N, the dimension of the space series, d_s, and the dimension of the time series, d_t. Our analysis indicates that for the one-dimensional CGLE model the average number of pulsed structures, N, grows according to a linear law with increasing dimension (fig. 5) in the region of amplitude turbulence as the parameter p is increased. The same linear growth is observed for the dispersion of the number of structures.

The dimensions d_s and d_t, for the general case, may be related arbitrarily. For instance, for a static disordered field distribution, $d_t = 0$ and $d_s > 2$. On the contrary, for a regular spatial field distribution with chaotic temporal dynamics, d_t may be rather large, while d_s is small. In a large box one-dimensional complex Ginzburg–Landau problem d_t grows with increasing effective length of the system (i.e. with increasing number of structures) and is much higher than the spatial dimension d_s. Within CGLE models that are described by specific dispersion laws there evidently exists a certain relation between d_s and d_t per unit length in the regime of established turbulence.

We would like to note that the jump of the dimension d_s in the motion over the parameter plane across the boundary of region 3 may occur a little later than that of d_t. This effect has a simple explanation. In the transition of the parameter across the critical value the intermittence of amplitude turbulence occurs: "holes" in the amplitude profile fluctuate, i.e. they disappear and appear at different moments of time. Therefore, if the space series corresponds to the moment, t, when there are no "holes", its dimension may be smaller that at other moments of time.

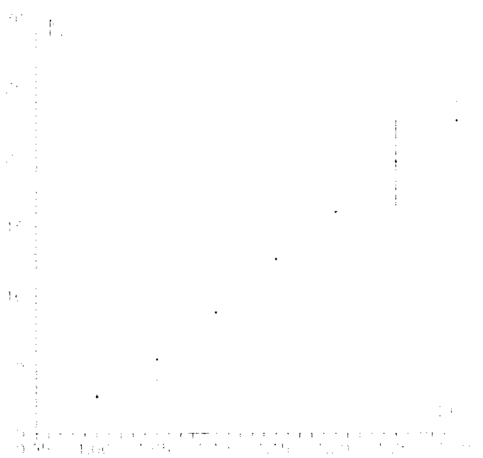

Fig. 5. Number of "holes", N, in amplitude turbulence against p for $R = 10^4$, $s = 1.15$. Vertical lines indicate the mean deviation of N.

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Appendix

The boundary value problem (2), (3) was solved numerically by the Fourier method. The integration algorithm was an apparent generalization of the method of operator exponents described in ref. [21]. Within this method exact solutions to a linear problem are employed to construct an approximate nonlinear solution, which shortens the computation time significantly as compared to standard grid algorithms. Spatial resolution was determined by the number of Fourier harmonics n. This number lies in the interval $512 < n < 8192$ and depends on the complexity of the regime and on the needed accuracy. Small sinusoidal amplitude perturbations or the spatial amplitude distributions that are established in a definite parameter region were taken as initial conditions.

The dimension of the space series in the solutions of interest can be estimated only if we have extended turbulent fields. However, insufficient computer memory and speed impede the calculations. Even for $n \geqslant 4096$ harmonics, an extremely long time is needed for the computation of solutions. So as to avoid this difficulty, especially in the case of very long space series, the following procedure was used.

The phase trajectory of the system in the established regime lies on an attractor and has stationary statistical properties. Consequently, we can use, for the determination of the dimension, space series taken at different moments of time. We match these series into one. To decrease the inhomogeneities at the matching points, we choose the series where the sequences of k neighboring points coincide and match them in the region of these sequences. The inhomogeneity in the region of matching can be neglected and the joint space series can be considered to be generated by a translational dynamical system, if the number k is larger than the dimension d_s on the stochastic set. In this fashion we can obtain arbitrarily long space series and determine the value of the dimension that is, actually, averaged over an ensemble of systems having different initial conditions.

The space resolution, Δx, in our numerical solutions was rather high to reveal the characteristic shape of the structures having the size l: $l/\Delta x \geqslant 10$. For the calculation of the correlation dynamical dimension, the stochastic set of a multidimensional dynamical system was reconstructed in a pseudo-phase space from the space series. A fixed-mass algorithm [22] and the Grassberger–Procaccia procedure [23] was employed to calculate the value d_s.

Proper "delay" of the space series is needed for the construction of the embedding space. If the "delay" step, ξ, is too small, the ensemble taken for the calculation of the correlation dimension will represent only the local structure of the solution (a section of a curve). The measured dimension will, consequently, tend to unity. If, on the other hand, the "delay" step is too large, the measured dimension will tend to the dimension of the embedding space since the points of the space series are completely uncorrelated.

A "delay" step ξ comparable with the correlation length provides a correct calculation of the dimension. A typical scale (for example, position of the first minimum) of the mutual information function gives the best choice for ξ [24]. The motivation for this is that the information in two successive delay coordinates should be as independent as possible, without making the delay too large. This criterion yields $\delta \simeq (3\text{–}5)l$. So as to verify this estimate and to determine the "delay" step more accurately we calculated the dependence of the dimension d_s on the "delay" step ξ. The function $d_s(\xi)$, as was to be expected, grows with increasing ξ (see fig. 6). The curve in the figure contains a flat section (a "plateau") which is the interval within which the "delay" step should be chosen. By choosing the value of ξ within this section we obtain a correct value of the dimension.

Besides, we calculated the dependence of d_s on the length of the space series N. The dimension d_s asymptotically tends to a constant that corresponds to the true dimension when $N \geqslant 2000$.

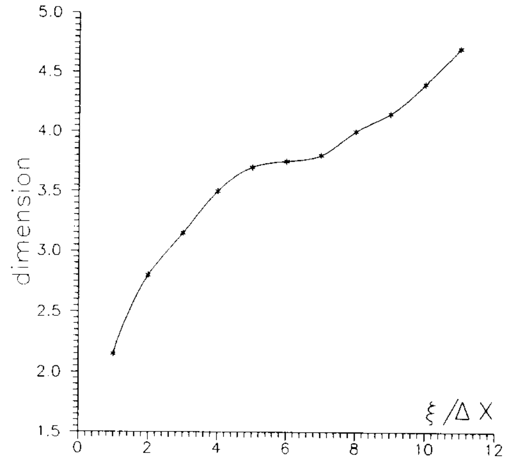

Fig. 6. The value of the dimension, d_s, calculated for various space delays ξ for $R=10^4$, $\alpha=1.0$, $\beta=1.3$. The number of harmonics used in the Fourier method is $n=2048$.

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