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The screening of a spiral field in a 2D complex Ginzburg–Landau equation

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The stability of one-dimensional dark strips, i.e. Nozaki–Bekki type solutions, relative to small and finite perturbations in a 2D complex Ginzburg–Landau equation is investigated analytically and numerically. It is shown that in a linear approximation in the region $-\infty<x<\infty$, $0\leqslant y\leqslant L$, Nozaki–Bekki strip (NBS) solutions are stable in a definite range of parameters. The regime of coexistence of NBS and chaotically walking spirals – bounded defect mediated turbulence – is found. In this regime the time averaged defect density in a rectangle bounded by parallel NBS solutions depends only on the value of supercriticality. Such a turbulent regime transforms, in a certain region of parameters, to a regime of spatio-temporal synchronization when the spirals are arranged periodically. The phenomenon of spiral field screening by a NBS field is shown. This phenomenon is responsible for spatial localization of turbulent regimes.

1. In the last few years regular and chaotic dynamics of localized defects in nonequilibrium media have been actively investigated in the literature (see, e.g., refs. [1–3]). One of the most consistent models for such an analysis is a CGLE describing oscillatory media near the Hopf bifurcation,

$$\partial_t a = a + (1+i\alpha)\nabla^2 a - (1+i\beta)|a|^2 a\,, \qquad (1)$$

where a is a complex value describing the phase and the amplitude of the field envelope (order parameter) while α and β are real parameters [4,5]. Most results have been obtained by computer analysis [1] but there are also analytical solutions, including an exact solution of the 1D CGLE in the form of a "dark soliton" or hole solution. This solution was first obtained by Nozaki and Bekki [6] [#1]. The explicit form of the hole solution is given by

$$a_1 = A(x,t)\exp[-i\omega t + i\theta(x)]\,, \qquad (2)$$

where $A(x,t)=\sqrt{1-Q^2}\tanh(kx)$, the phase θ fulfills the equation

$$\frac{\partial\theta}{\partial x} = -Q\tanh(kx)\,, \qquad (3)$$

where $\omega=(\alpha-\beta)Q^2+\beta$ is the frequency, and $Q=(2k^2-1)/3k\alpha$ is the wavenumber. The parameter $1/k$ describes the width of the hole and k is found from the equation

$$[4(\beta-\alpha)+18\alpha(1+\alpha^2)]k^4 - [4(\beta-\alpha)+9\alpha(1+\alpha\beta)]k^2 + (\beta-\alpha) = 0\,. \qquad (4)$$

Solution (2) tends asymptotically, as $x\to\pm\infty$, to a plane wave solution with wavenumber $\mp Q$. The amplitude of the hole solution turns to zero when $x=0$. The dynamics of the one-dimensional defects (2) considered here has been studied in ample detail (see, e.g., refs. [8–12]).

An isolated spiral solution,

$$a_2 = F(r)\exp[-i\omega t - im\varphi + i\theta(r)]\,, \qquad (5)$$

is most frequently encountered in analyses of 2D CGLE. Here (r,φ) are polar coordinates, m is a topological charge (only solutions with $m=\pm1$ are stable [13]), $\omega=(\alpha-\beta)Q^2+\beta$ is the frequency of rotation, and Q is an asymptotic wavenumber that

[#1] An analogous solution for a conservative system, a nonlinear Schrödinger equation, was obtained by Hasegawa and Tappert in 1973 [7].

depends on α and β [13]. The functions $F(r)$ and $\theta(r)$ may be represented for $r \gg 1$ in asymptotic forms,

$$F(r)=(1-Q^2)^{1/2}-\frac{(1+\alpha^2)Q}{2(1-Q^2)^{1/2}(\alpha-\beta)}r^{-1}+\mathrm{O}(r^{-2}), \quad (6)$$

$$\theta(r)=Qr+\frac{1+\alpha\beta}{2(\alpha-\beta)}\ln(r)+\mathrm{O}(r^{-1}). \quad (7)$$

Much attention has also been given to the investigation of the stability and interaction of spirals [14–16].

In spite of the substantial difference between the holes (2) and the spirals (5), their behaviors as point defects in oscillatory media have much in common and from this point of view solution (5) is often considered as a two-dimensional analog of (2).

In this paper we will show that solution (2) may be directly generalized to the case of a 2D medium and the generalized solution, i.e. the NBS, will be stable. We will also demonstrate by means of a computer experiment fascinating phenomena associated with the interaction of the NBS patterns and spirals: screening of the spiral field by NBS, localization of defect mediated turbulence, as well as space and time synchronization of spirals by a NBS field.

2. Consider a two-dimensional Ginzburg–Landau equation in the region $\Omega=\{-\infty<x<\infty,\ 0\leqslant y\leqslant L\}$. Periodic boundary conditions are chosen along the spatial coordinate y. Then it is apparent that the Nozaki–Bekki hole solution (2) of a one-dimensional CGLE will also be a solution of a two-dimensional equation uniform along the y-coordinate. Thus, in what follows we will be interested in the solution of eq. (1) in the 2D case in form (2) (NBS solution).

First of all we will show that solution (2) is linearly stable with respect two-dimensional perturbations of the field. The evolution of a small perturbation $\tilde{a}(x, y, t)$ of amplitude $A(x, t)$ of the solution $a_1(x, t)$ to eq. (1) is described by a linear equation,

$$\partial_t\tilde{a}=\Lambda(\tilde{a}, \tilde{a}^*, x)+(1+\mathrm{i}\alpha)\partial_y^2\tilde{a}, \quad (8)$$

where

$$\Lambda(\tilde{a}, \tilde{a}^*, x)=\{(1+\mathrm{i}\omega)+(1+\mathrm{i}\alpha)[\partial_x^2+2\mathrm{i}\partial_x\theta\,\partial_x-(\partial_x\theta)^2+\mathrm{i}\partial_x^2\theta]-2(1+\mathrm{i}\beta)|A|^2\}\tilde{a}-(1+\mathrm{i}\beta)A^2\tilde{a}^*. \quad (9)$$

By separating in eq. (8) the real and the imaginary parts ($\tilde{a}=u+\mathrm{i}v$) we obtain the following system,

$$\partial_t\begin{pmatrix}u\\v\end{pmatrix}=(\tilde{\Lambda}+S)\begin{pmatrix}u\\v\end{pmatrix},\quad S=\partial_y^2\begin{pmatrix}1&-\alpha\\\alpha&1\end{pmatrix}, \quad (10)$$

where $\tilde{\Lambda}$ is a linear operator that depends only on the spatial coordinate x.

We will seek solutions of (10) in the form

$$\begin{pmatrix}u(x,y,t)\\v(x,y,t)\end{pmatrix}=\sum_{n=-\infty}^{\infty}\begin{pmatrix}U_n(x,t)\\V_n(x,t)\end{pmatrix}W_n(y), \quad (11)$$

where $W_n(y)$ is the eigenfunction of operator ∂_y^2 in the region $y\in[0, L]$ corresponding to the eigenvalue p_n. For example, $W_n(y)=\exp(\mathrm{i}k_ny)$, $k_n=2\pi n/L$, $p_n=-k_n^2$ for periodic boundary conditions. For the amplitudes $U_n(x, t)$ and $V_n(x, t)$ we have the following equation,

$$\partial_t\begin{pmatrix}U_n\\V_n\end{pmatrix}=\tilde{\Lambda}\begin{pmatrix}U_n\\V_n\end{pmatrix}+p_n\begin{pmatrix}1&-\alpha\\\alpha&1\end{pmatrix}\begin{pmatrix}U_n\\V_n\end{pmatrix}. \quad (12)$$

It should be noted that this region of the hole stability (2) within the CGLE in a one-dimensional case contains an interval $\beta\in(\beta_1, \beta_2)$ for $\alpha=0$ (see refs. [9,10]). At the same time, it is clear from (12) that for $\alpha=0$ the second term in the right-hand side of eq. (12) shifts the spectrum of the operator $\tilde{\Lambda}$, $\sigma(\tilde{\Lambda})$, along the real axis to the left ($p_n=-k_n^2<0$) and, consequently, $\sigma(\tilde{\Lambda}+S)$ belongs to the left half-plane, given that $\sigma(\tilde{\Lambda})$ lies to the left of the real axis. This statement is also valid for sufficiently small values of α by virtue of the continuous dependence of the operator spectrum in the right-hand side of (12) on the parameter α.

Thus, we can contend that the NBS solution is stable relative to 2D perturbations in a certain region of the (α, β) plane. The problem of stability of the NBS solution of the CGLE in a 2D case for sufficiently small values of the parameter α reduces to the problem of stability of solution (2) of the CGLE in a 1D case. The latter problem was broadly discussed in the literature. We can refer the reader, for instance, to the results of refs. [8–10], where the stability region of the hole solution for the 1D CGLE was constructed on the (α, β) plane.

3. Numerical experiments have demonstrated the stability of the NBS in a two-dimensional medium relative to finite perturbations. We employed for the integration of eq. (1) a pseudo-spectral method [17] based on FFT with periodic boundary conditions. The region of integration had the size 150×150, the number of FFT harmonics was taken to be 256×256 or 512×512, and the integration step was approximately 0.1.

The most important, and to a certain extent unexpected, result is the following: nonlocalized NBS are not destroyed by 2D patterns of finite amplitude and may coexist, in particular, with spirals for rather long times $T\geqslant10^4$, their mutual dynamics depending significantly on the choice of the parameters α and β.

The level lines for the amplitude and phase of the field snapshot at $\beta=2$ and $\alpha=0.2$ are presented in fig. 1. In the presence of NBS, the preset "spot" containing spirals chaotically arranged in space evolves rather fast into a sequence of spirals arranged along one line at almost equal distances from one another. Further, the topology of this established field distribution remains unchanged throughout the integration interval $T\approx10^4$.

Figure 2 shows the field distribution that contains only the NBS solution and is stable at the same values of the parameters $\beta=2$ and $\alpha=0.2$. The longitudinal structure of the field is given in fig. 3 for a certain fixed value of $y=$const. It is typical for the solution of a 1D Ginzburg–Landau equation (see, e.g., ref. [8]). The holes and the shocks separating them are well pronounced in the picture. Thus, in the case of interest not only individual nonlocalized solutions but also the field as a whole, including shocks, have a structure uniform along the spatial coordinate y.

For other values of the parameters, the spirals are never regularly arranged in the presence of NBS. The snapshot of the field shown in fig. 4 is for $\beta=2$, $\alpha=0.1$ (as the starting field distribution we chose the one given in fig. 2). The field evolves to this state as follows. First, additional NBS are formed in the medium, with the uniform field structure persisting along y (i.e. the medium behaves as one-dimensional). Then part of the structures are destroyed (see the snapshot in fig. 5) giving birth of numerous spirals with topological charges of different signs. A little later, a spatial distribution is established which contains several nonlocalized NBS slowly drifting along the x-axis and a family of spirals whose coordinates vary chaotically in time. An analogous situation is depicted in fig. 6 where the spirals are localized only in one region between NBS (this

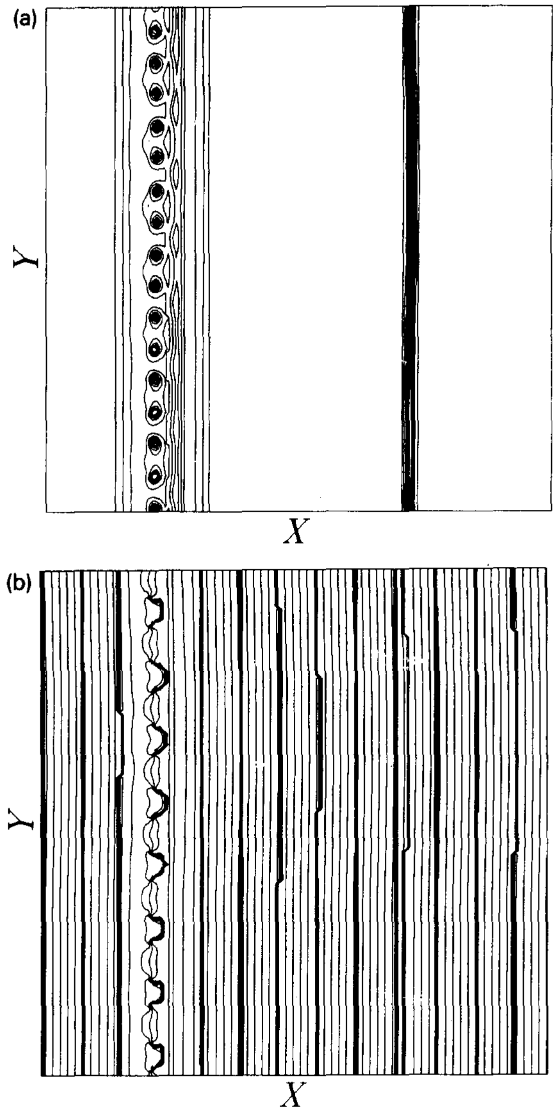

Fig. 1. Level lines of amplitude (a) and phase (b) of an established field distribution for $\beta=2$, $\alpha=0.2$ and $T=10^4$. A regular distribution of the spirals demonstrates the effect of spatial self-organization under random initial conditions.

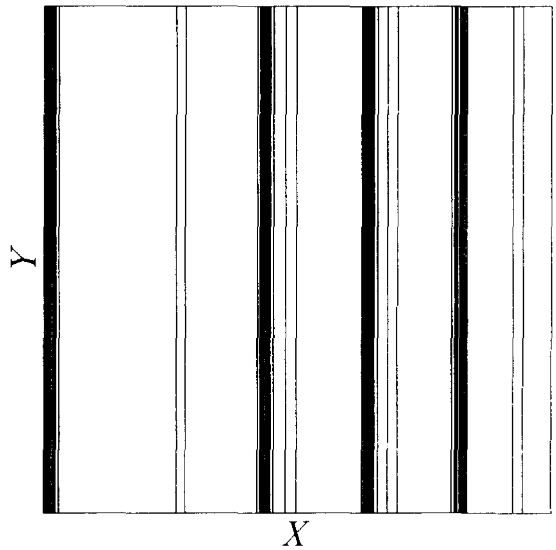

Fig. 2. Level lines of the amplitude of an established field distribution. The parameters are the same as in fig. 1 but the initial conditions are different. A comparison of figs. 1 and 2 shows that finite perturbations are needed for the birth of spirals.

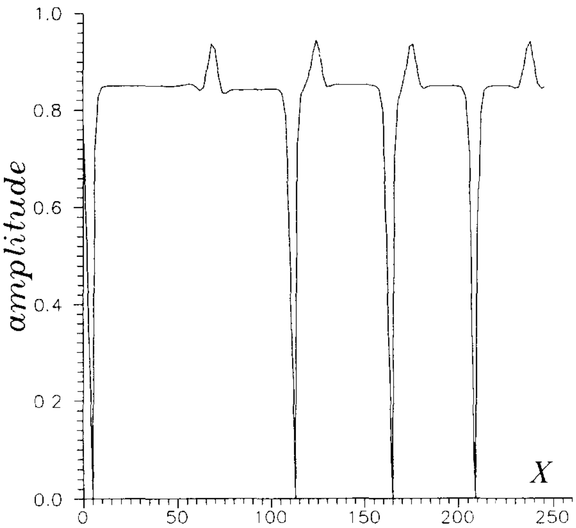

Fig. 3. Longitudinal structure of the field at $y=0$ corresponding to the snapshot in fig. 2.

snapshot of the field was taken at other initial conditions).

Note that if the field distribution presented in fig. 4 is taken as the initial one and the equation is integrated at $\beta=2$, $\alpha=0.2$, then the systems comes rather fast to the regular form shown in fig. 7.

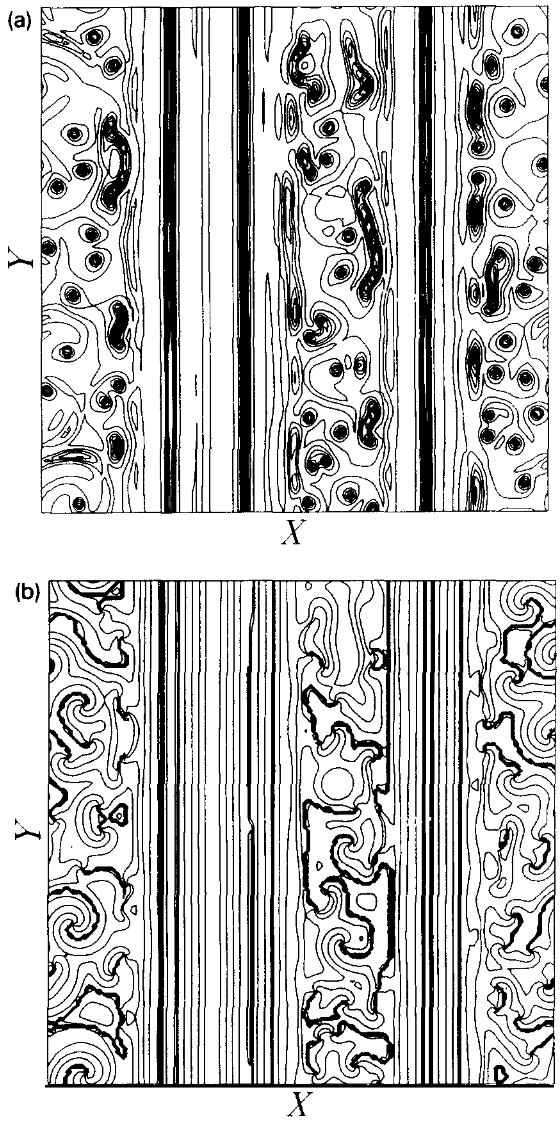

Fig. 4. Level lines of amplitude (a) and phase (b) of an established field distribution for $\beta=2$, $\alpha=0.1$ and $T=10^4$. The field distribution presented in fig. 2 is taken as initial condition. For these parameter values additional NBS are born in the initial medium, then part of them are destroyed giving rise to defect mediated turbulence.

Thus, depending on the values of the parameters α and β, we can distinguish two principal regimes of coexistence of NBS and spirals. In the first case the spiral dynamics is completely suppressed by the field of nonlocalized NBS and the spirals are arranged

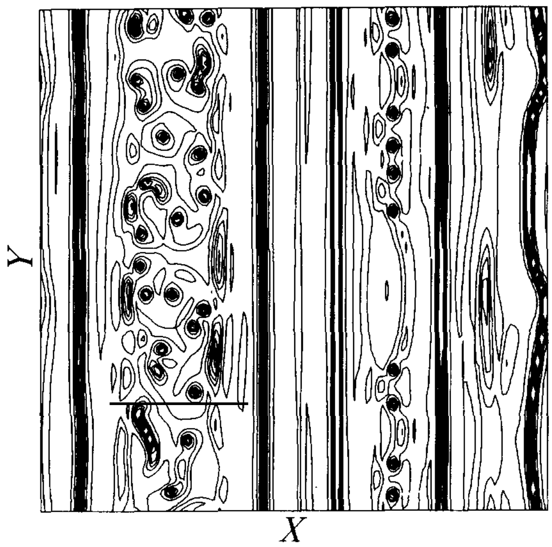

Fig. 5. Level lines of amplitude of a developing ($T=3\times10^3$) field distribution for $\beta=2$, $\alpha=0.1$ and initial conditions as in fig. 2. A NBS deformed in the transverse direction is observed in the right-hand side of the box at the moment preceding destruction.

along one line at a maximal distance from these structures. In the second case the spirals retain their own dynamics and move chaotically in space between the neighboring nonlocalized NBS. It is essential that the NBS screen the neighboring region from the action of spirals.

The plane of the parameters α, β containing regions of different dynamics is shown in fig. 8 [18,19], where the values of the parameters at which the phenomena of interest were observed are marked.

4. One of the most remarkable consequences of the effects observed is that a bounded turbulent region may exist in an unbounded (along x) strip of the nonequilibrium medium described by the CGLE. Actually, this means that a finite-dimensional strange attractor may exist in a system with an infinite Reynolds number.

It is not clear yet how the different NBS interact with one another and whether they may be mutually synchronized (when the radiated waves are in phase). The problem of the intrinsic structure of the NBS also remains open. Numerical experiments show that the structure of the NBS at strong transverse distortions looks like a chain of coupled, closely placed localized field singularities of spiral pair type.

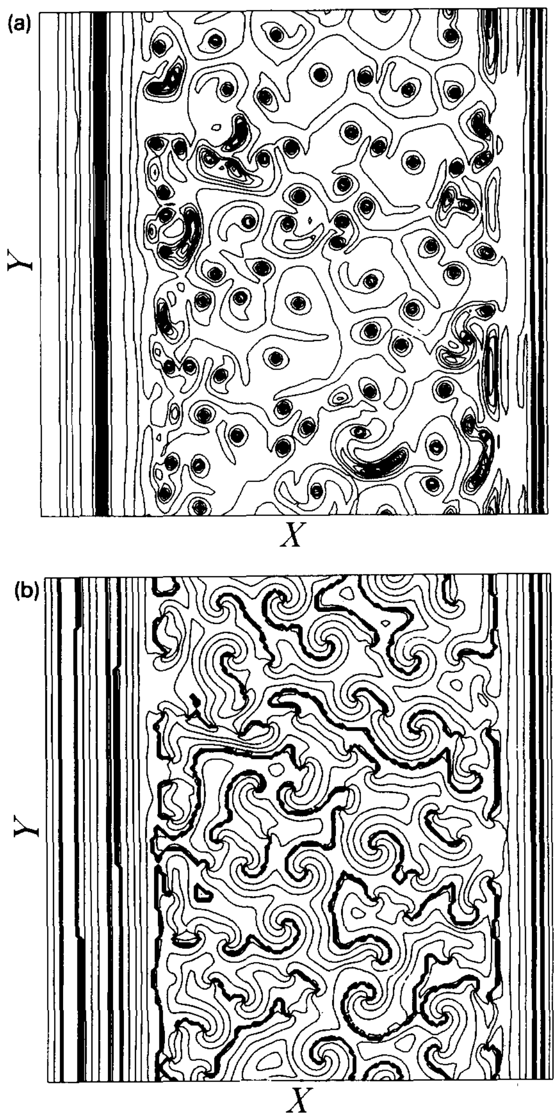

Fig. 6. Level lines of amplitude (a) and phase (b) of an established field distribution for $\beta=2$, $\alpha=0.1$ and $T=10^4$. The distribution containing a pair of NBS and a few randomly located spirals is taken as initial condition. The density of spirals depends only on the parameters (cf. fig. 4).

Finally, it is extremely important to clarify the role of the boundary conditions along y and their effect on the stability of the NBS.

It is not excluded, of course, that part of the nonlocalized structures observed in the experiment will

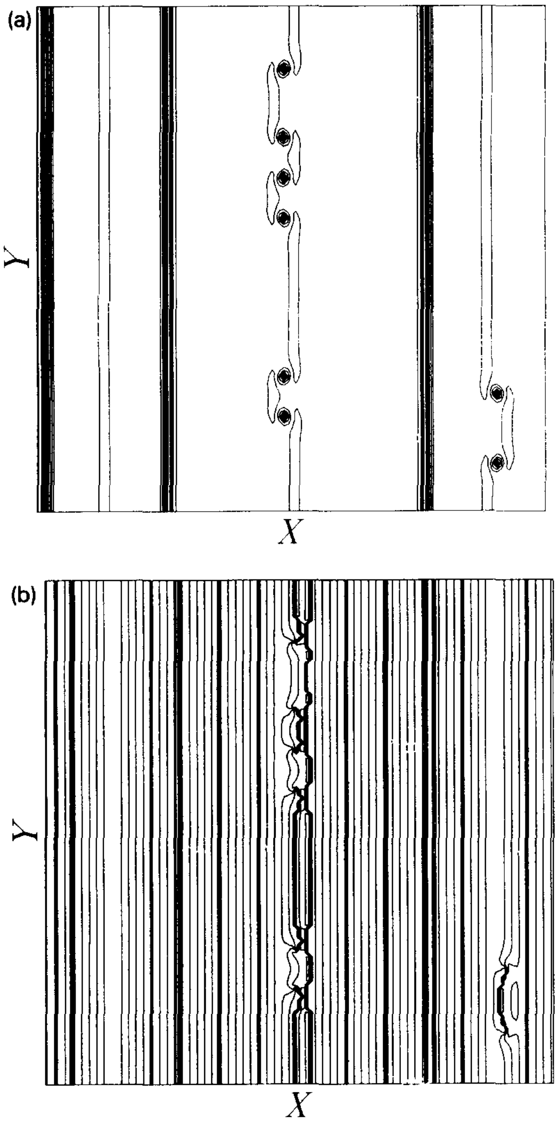

Fig. 7. Level lines of amplitude (a) and phase (b) of an established field distribution for $\beta=2$ and $\alpha=0.2$. The field distribution depicted in fig. 4 is taken as initial condition. This is still another confirmation of the phenomenon of spatial self-organization.

disintegrate into separate spirals at very large times ($T_\infty > 10^4$). However, this does not remove the observed effects because the system does not contain any characteristic times close to T_∞. Besides, as was shown in ref. [21], the hole solutions moving with a slow velocity may be stabilized in computer sim-

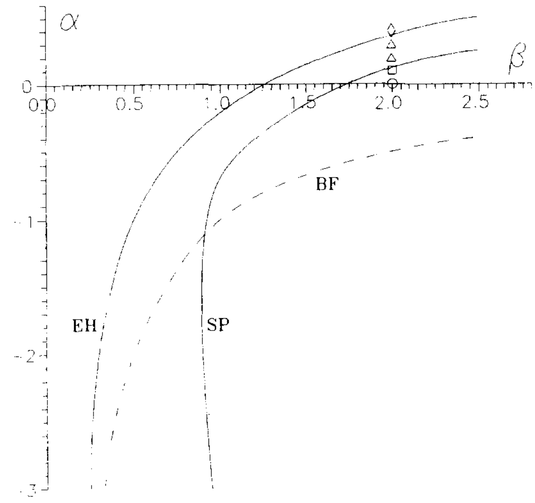

Fig. 8. The plane of the parameters α and β. Presented are the Benjamin–Fair limit (dotted line, BF); the long wavelength Eckhaus limit with $Q(\alpha, \beta)$ corresponding to 2D spirals (solid line, EH); and the absolute stability limit for 2D spirals (solid line, SP) [18]. Marked are the values of the parameters corresponding to the regimes observed: (○) developed defect mediated turbulence, NBS are destroyed; (□) defect mediated turbulence in the region between stable NBS (see figs. 4, 6); (△) spatial self-organization of spirals in the region between stable NBS (see figs. 1, 7); (◇) quasistationary spatial disorder of spirals, NBS are destroyed.

ulation of the CGLE. This may occur, in particular, due to an additional small term in (1) of the form $-\epsilon |a|^4 a$ generated by the numerical scheme.

We would like to add that the phenomenon of turbulent localization in a bounded region was obtained in experiments on bimodal convection with large Prandtl numbers. The boundary between the turbulent and the coherent regions was unstable but very long-lived [20].

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